Worcester County Mathematics League

Varsity Meet 1
October 9, 2013

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Varsity Meet 1 – October 9, 2013
Round 1: Arithmetic

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If $a \star b = \dfrac{a^2 + ab}{a + b}$, evaluate $\dfrac{1}{2} \star \dfrac{2}{3}$.

2. Evaluate:

$$\frac{2\frac{1}{3}}{21} \times \frac{3}{5} \div \left(1\frac{1}{2} \times 0.4\right)$$

3. Let $A \triangle B = \dfrac{A - B}{AB - A}$ and $A \diamond B = \left(2A + \dfrac{A}{B}\right) \triangle B^2$. Evaluate $4 \diamond 2$.

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

Varsity Meet 1 – October 9, 2013
Round 2: Algebra I

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. What must be added to $3x^2 + 7x - 5$ to give $7x^2 - 2x + 6$? Give your answer in standard polynomial form.

2. Solve for all possible values of x:

$$\frac{24}{x+2} + \frac{12}{x} = 5.$$

3. During a race upstream and back, 60 miles in each direction, a boat averages 18 mph for the round trip. The boat can travel 24 mph in still water. Assume a constant current over the duration of the race. In miles per hour, what is the rate of the current?

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. $x =$ ____________________

(3 pts.) 3. ____________________ mph

Varsity Meet 1 – October 9, 2013
Round 3: Set Theory

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Let

$$A = \{2, 3, 6, 7, 9\},$$
$$B = \{0, 1, 2, 3, 4, 5\}, \text{ and}$$
$$C = \{2, 4, 6\}.$$

Find $(A \cap B) \cup (C \cap A)$.

2. The universal set for this problem is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. Let $D = \{1, 3, 7, 11\}$, $E = \{2, 5, 7, 9, 11\}$, and $F = \{4, 7, 11\}$. If S' denotes the complement of S, find

$$(D \cup E)' \cap (E' \cap F').$$

3. Set P is a subset of set Q. P has 3 elements and Q has 10. How many subsets of Q completely contain set P?

ANSWERS

(1 pt.) 1. { ________________ }

(2 pts.) 2. { ________________ }

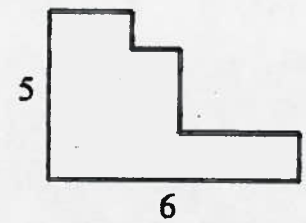
(3 pts.) 3. ________________

Varsity Meet 1 – October 9, 2013
Round 4: Measurement

All answers must be in simplest exact form in the answer section

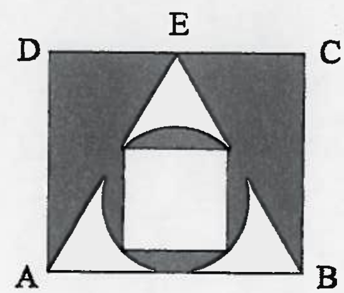
NO CALCULATOR ALLOWED

1. Given that all intersecting segments in the polygon shown are perpendicular, what is the perimeter of the figure?



2. An equilateral triangle and a regular hexagon have equal perimeters. Find the ratio of the area of the triangle to the area of the hexagon and express it in the form $T : H$, with T and H relatively prime.

3. In the figure, $\triangle ABE$ is equilateral with perimeter 18 and is inscribed in rectangle $ABCD$. A circle is inscribed in $\triangle ABE$ and a square inscribed in that circle. Find the exact area of the shaded region.



ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. __________ : __________

(3 pts.) 3. ____________________

Varsity Meet 1 – October 9, 2013
Round 5: Polynomial Equations

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If -3 is one root of $8x^2 + ax - 15 = 0$, what is the value of a?

2. If i represents the imaginary unit $\sqrt{-1}$, what is the ordered pair of real numbers (a, b) for which

$$(1+i)^{13} = a + bi\,?$$

3. Suppose a given polynomial $p(x)$ leaves a remainder of 1 upon division by $x - 1$ and a remainder of 5 upon division by $x - 3$. What is the remainder when p is divided by $(x-1)(x-3)$? Give your answer is standard polynomial form.

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. (__________ , __________)

(3 pts.) 3. ______________________

Varsity Meet 1 – October 9, 2013
TEAM ROUND

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (2 POINTS EACH)

APPROVED CALCULATORS ALLOWED

1. Let A, B, and C be sets. If $A \cup C = \{1,2,3,4,5,6\}$, $B \cup C = \{1,2,3,4\}$, $A \cap C = \varnothing$, $A \cap B = \{3\}$, and $B \cap C = \{1,2\}$, find B.

2. Consider the Cartesian plane and let X denote the subset of points for which both coordinates are integers (*i.e.* LATTICE POINTS). A circular coin of diameter $1/2$ is tossed randomly onto the plane. Find the exact probability that the coin covers a point in X.

3. Let a, b, c, d be real coefficients. Find the product $abcd$ if
$$(ax+b)(cx+d) = (2x+1)^3 - 4x(x+3)(1+2x).$$

4. Albert E. pays \$8.00 for a square piece of polystyrene, which he makes into a stop sign (regular octagon) by cutting off the corners. To the nearest cent, what is the cost of the wasted part?

5. Find the coefficient of the x^2 term in the expansion of $(x^3 - a/x)^{10}$.

6. It takes 732 digits to number the pages of a book consecutively, starting with page 1. How many pages are in the book?

7. Tickets for a concert were \$24 for the main floor and \$19 for the balcony. If the receipts from the sale of 1600 tickets totaled \$35170, how many tickets were sold for the main floor?

8. When the radius of a circle is increased by 5, its area is increased by 32π. Find the radius of the original circle.

9. If $a \downarrow b = \sqrt{\dfrac{ab}{a-b}}$, find the value of $\dfrac{8 \downarrow 6}{3 \downarrow 2}$.

Varsity Meet 1 – October 9, 2013
TEAM ROUND ANSWER SHEET

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (2 POINTS EACH)

1. { ________________ }

2. ________________

3. ________________

4. $ ________________

5. ________________

6. ________________ pages

7. ________________ main floor tickets

8. ________________

9. ________________

Varsity Meet 1 – October 9, 2013
ANSWERS

ROUND 1
(Bancroft, Tahanto, Leicester)

1. $1/2 = 0.5$
2. $1/9 = 0.\overline{1}$
3. $1/5 = 0.2$

ROUND 2
(Leicester, Auburn, Doherty)

1. $4x^2 - 9x + 11$
2. $x = 6, -4/5$ (either order)
3. 12 mph

ROUND 3
(Bartlett, Bartlett, Worc Acad)

1. $\{2, 3, 6\}$
2. $\{6, 8, 10\}$
3. 128

ROUND 4
(Hudson, Tahanto, Worc Acad)

1. 22
2. $2:3$
3. $9\sqrt{3} + 3\pi - 6$ (or equivalent, based on the commutative and/or associative property)

ROUND 5
(Assabet Valley, Hudson, QSC)

1. 19
2. $(-64, -64)$
3. $2x - 1$

TEAM ROUND
(Athol, Doherty, Worc Acad, Burncoat, Tantasqua, Doherty, Millbury, Shrewsbury, Doherty)

1. $\{1, 2, 3\}$
2. $\pi/16$
3. -16
4. \$1.37
5. $-120a^7$
6. 280 pages
7. 954 main floor tickets
8. $7/10 = 0.7$
9. 2

Varsity Meet 1 – October 9, 2013
FULL SOLUTIONS

ROUND 1

1. We have that $a \star b = \dfrac{a^2+ab}{a+b} = \dfrac{a(a+b)}{a+b} = a$ for $a+b \neq 0$. Therefore, $\dfrac{1}{2} \star \dfrac{2}{3} = \boxed{\dfrac{1}{2}}$.

2. Following order of operations,

$$\begin{aligned}\frac{2\frac{1}{3}}{21} \times \frac{3}{5} \div \left(1\frac{1}{2} \times 0.4\right) &= \frac{7/3}{21} \times \frac{3}{5} \div \left(\frac{3}{2} \times \frac{2}{5}\right)\\ &= \frac{1}{9} \times \frac{3}{5} \div \frac{3}{5}\\ &= \boxed{\frac{1}{9}}.\end{aligned}$$

3. Substituting the definitions,

$$\begin{aligned}4 \diamond 2 &= \left(2 \cdot 4 + \frac{4}{2}\right) \triangle 2^2\\ &= 10 \triangle 4\\ &= \frac{10-4}{10 \cdot 4 - 10}\\ &= \frac{6}{30}\\ &= \boxed{\frac{1}{5}}.\end{aligned}$$

ROUND 2

1. Subtract: $(7x^2 - 2x + 6) - (3x^2 + 7x - 5) = \boxed{4x^2 - 9x + 11}$.

2. First clear fractions, and then solve the quadratic.

$$\begin{aligned}\frac{24}{x+2} + \frac{12}{x} &= 5\\ 24x + (12x + 24) &= 5(x+2)(x)\\ 0 &= 5x^2 - 26x - 24\\ 0 &= (5x+4)(x-6).\end{aligned}$$

Therefore, $x = \boxed{-4/5, 6}$. Neither solution is extraneous.

3. The distance does not matter, as long as the upstream and downstream distances are equal. The total time can be calculated in two ways: as the sum of each direction, and as the total from the overall average speed. Equating these two different calculations will give the speed of the current.

$$\begin{aligned}
\frac{60}{24-x}+\frac{60}{24+x} &= \frac{120}{18} \\
\frac{1}{24-x}+\frac{1}{24+x} &= \frac{1}{9} \\
\frac{48}{24^2-x^2} &= \frac{1}{9} \\
-x^2+24^2 &= 9\cdot 48 \\
9(64-48) &= x^2 \\
9\cdot 16 &= x^2 \\
3\cdot 4 &= x \\
\boxed{12} &= x
\end{aligned}$$

ROUND 3

1. We have that $(A\cap B)\cup(C\cap A)=\{2,3\}\cup\{2,6\}=\boxed{\{2,3,6\}}$.

2. We have

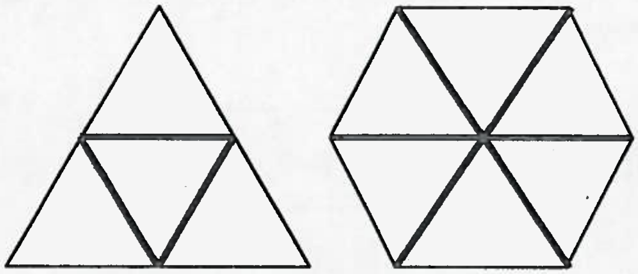
$$\begin{aligned}
& (D\cup E)'\cap(E'\cap F') \\
= \ & \{4,6,8,10\}\cap\left[\{1,3,4,6,8,10\}\cap\{1,2,3,5,6,8,9,10\}\right] \\
= \ & \{4,6,8,10\}\cap\{1,3,6,8,10\} \\
= \ & \boxed{\{6,8,10\}}.
\end{aligned}$$

3. Such a set must contain the three elements in P. It can also contain any number of the remaining 7 elements in Q that are not in P. There are $2^{10-3}=\boxed{128}$ such sets.

ROUND 4

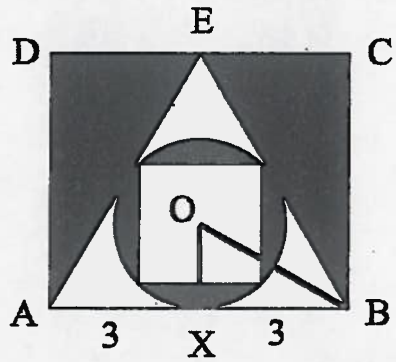
1. The concave parts can be "pushed out" without affecting the perimeter to form a 5×6 rectangle. The perimeter is $2(5+6)=\boxed{22}$.

2. Divide the triangle and hexagon into congruent equilateral triangles. Each has a perimeter equal to six side lengths.



From this diagram, it is easy to see that the ratio of areas is $4 : 6 = \boxed{2 : 3}$.

3. Let O be the center of the inscribed circle and square and X the midpoint of $\overline{AB}$, as shown below:



Since $\triangle ABE$ has perimeter 18, $AX = XB = 3$. The sum of the areas of the shaded triangles is equal to the area of $\triangle ABE$, which is $\frac{6^2\sqrt{3}}{4} = 9\sqrt{3}$. Additionally, $\triangle OBX$ is a 30-60-90 right triangle, so $OX = \sqrt{3}$. Therefore, the area of the circle is 3π. The inscribed square has diagonal $2\sqrt{3}$, so its area is $d^2/2 = 6$. Hence the total area of the shaded region is $\boxed{9\sqrt{3} + 3\pi - 6}$.

ROUND 5

1. Plug in $x = -3$ and solve for a:

$$\begin{aligned} 8 \cdot 9 - 3a - 15 &= 0 \\ 57 &= 3a \\ \boxed{19} &= a \end{aligned}$$

2. **METHOD I**: Note that $(1+i)^2 = 1+2i-1 = 2i$. Therefore, $(1+i)^{13} = (1+i)[(1+i)^2]^6 = (1+i)(2i)^6 = (1+i)\cdot -64 = -64-64i$. In the form $a+bi$, $(a,b) = \boxed{(-64,-64)}$.

   METHOD II: Using polar coordinates[1], $1+i = \sqrt{2}e^{i\pi/4}$. Therefore, $(1+i)^{13} = (\sqrt{2})^{13}e^{13i\pi/4} = 64\sqrt{2}e^{5i\pi/4} = 64(-1-i) = -64-64i$, as before.

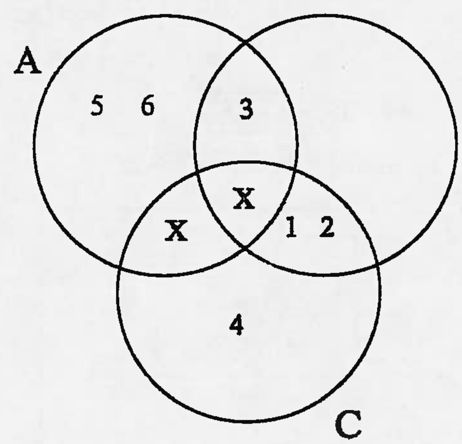
3. By the REMAINDER THEOREM, we know that $p(1) = 1$ and $p(3) = 5$. The polynomial p can be written as a quotient q and remainder r:

$$p(x) = q(x)(x-1)(x-3) + r(x).$$

   Therefore, we know that $p(1) = 1 = r(1)$ and $p(3) = 5 = r(3)$, since in each of those two cases the q term is multiplied by zero. Also, since $\deg[(x-1)(x-3)] = 2$, the remainder has degree at most $2-1 = 1$. Let the remainder be $r(x) = ax+b$. Then, $r(1) = a+b = 1$ and $r(3) = 3a+b = 5$. Solving simultaneously, $a = 2$ and $b = -1$, so the remainder is $\boxed{2x-1}$.

TEAM ROUND

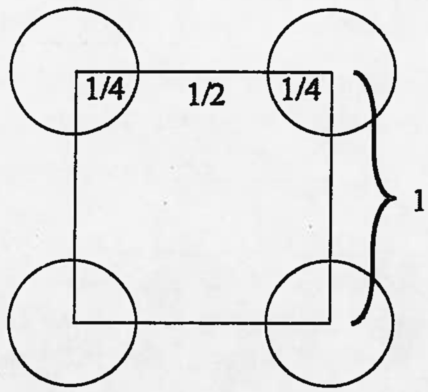
1. Start with the fact that $A \cap C = \varnothing$. Then, place the values of the other intersections $A \cap B$ and $B \cap C$. Finally, place 4, 5, and 6 to satisfy the unions.



   Therefore, $B = \boxed{\{1,2,3\}}$.

[1]You may have seen polar coordinates written in the form r cis θ, where r is the magnitude of the vector and θ is the angle it makes with the positive x-axis. The "cis" notation is just shorthand for $\cos\theta + i\sin\theta$, which by EULER'S FORMULA is equal to $e^{i\theta}$ for θ in radians.

2. Keep track of the position of the center of the coin. Let the domain be a 1×1 unit square. If the center of the coin falls inside any of the sectors, some portion of the coin will cover that lattice point.



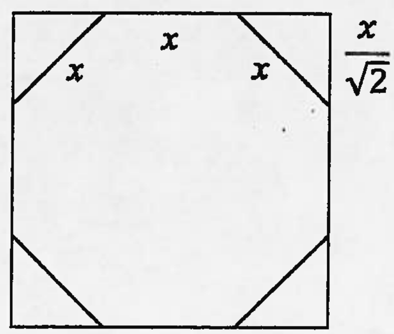
The total area of the four sectors is $\pi/16$, and the area of the domain is 1, so the probability is $(\pi/16)/1 = \boxed{\pi/16}$.

3. Factor the expression into the form $(ax+b)(cx+d)$:

$$\begin{aligned}(2x+1)^3 - 4x(x+3)(1+2x) &= (2x+1)[(2x+1)^2 - 4x(x+3)] \\ &= (2x+1)(4x^2+4x+1-4x^2-12x) \\ &= (2x+1)(-8x+1).\end{aligned}$$

Therefore, $abcd = (2)(1)(-8)(1) = \boxed{-16}$.

4. Let the side length of the regular octagon be x.



The wasted portions are 45-45-90 right triangles, so the total side length of the square is $(1+\sqrt{2})x^2$. The total area of the four wasted triangles is x^2. Therefore, the value is

$$\frac{1}{(1+\sqrt{2})^2} \times \$8.00 = \boxed{\$1.37}.$$

5. To get x^2, we need three factors of x^3 and seven factors of $-a/x$. Therefore, the numerical coefficient is $\binom{10}{3} = 120$. The sign of the coefficient is $(-a)^7 = -a^7$, so the total coefficient is $\boxed{-120a^7}$.

6. It requires 9 digits to enumerate the pages 1–9 and 180 digits for the ninety pages from 10–99. This leaves $732 - 189 = 543$ more digits for three-digit numbers. That is enough for $543/3 = 181$ more pages, starting with page 100. Therefore, there are $181 + 99 = \boxed{280}$ total pages in the book.

7. **METHOD I:** If all tickets were for the balcony, the total value would be $1600 \cdot \$19 = \$30400$. This is too low by $\$35170 - \$30400 = \$4770$. Since a main floor ticket costs \$5 more than a balcony ticket, there were $\$4770 \div \$5 = \boxed{954}$ main floor tickets sold.

   METHOD II: Solve the 2×2 system of equations. Let m be the number of main floor tickets and b be the number of balcony tickets sold. Then:

$$\begin{aligned} 24m + 19b &= 35170 \\ m + b &= 1600 \end{aligned}$$

   From the second equation, $b = 1600 - m$, so $24m + 19(1600 - m) = 35170$. Thus $5m + 19 \times 1600 = 35170$ and $m = \boxed{954}$, as before.

8. Since the area of a circle is πr^2, we have that

$$\begin{aligned} (r+5)^2 &= r^2 + 32 \\ 10r + 25 &= 32 \\ 10r &= 7 \\ r &= \boxed{7/10}. \end{aligned}$$

9. Plug in:

$$\begin{aligned} \frac{8 \downarrow 6}{3 \downarrow 2} &= \frac{\sqrt{48/2}}{\sqrt{6}} \\ &= \sqrt{\frac{8}{2}} \\ &= \boxed{2}. \end{aligned}$$